Self-interaction near dielectrics

Lior M. Burko
Department of Physics, University of Utah, Salt Lake City, Utah 84112
(Received 4 December 2001; published 10 April 2002)

We compute the force acting on a free, static electric charge outside a uniform dielectric sphere. We view this force as a self-interaction force, and compute it by applying the Lorentz force directly to the charge's electric field at its location. We regularize the divergent bare force using two different methods: A direct application of the Quinn-Wald comparison axiom and mode-sum regularization.

DOI: 10.1103/PhysRevE.65.046618 PACS number(s): 41.20.−q, 41.60.−m

I. INTRODUCTION

Electric charges that are placed in an inhomogeneous ponderable medium undergo self-interaction. The simplest case is that of a static electric charge in an inhomogeneous dielectric. The self interaction of the charge results in a self-force (in other contexts this self-force is also known as a radiation reaction force), which acts to accelerate the charge. In the static problem, then, one can ask the following question: What is the external, possibly nonelectric, force that needs to be exerted on the charge to keep it static when a nonuniform dielectric is present?

In this paper we study this question for a very simple case. Specifically, we find the self force on a pointlike electric charge e outside a uniform dielectric sphere. (By an inhomogeneous dielectric we here mean the discontinuity of the dielectric constant at the surface of the sphere.) The origin of the force on the charge in this case is simple: The charge polarizes the dielectric at order e. The induced electric field then backreacts on the original charge, and this interaction then is at order e^2. In fact, one can compute the force on the charge following this simple physical picture. However, one can employ a different picture, in which the force on the charge is construed as a self-force. The charge interacts with its own field, and the latter is distorted by the presence of the dielectric sphere. In this picture the force is computed *locally*, using only the fields at the location of the charge. The local approach has many merits. Specifically, the computation of the near field is much simpler, and one does not have to compute additional quantities such as the far field or the sphere's polarization. (The fields at great distances of course contribute to the force on the charge, but for the local approach only through boundary conditions.) A similar approach was used in Ref. [1] to compute the radiated power of synchrotron radiation using only the near field. The difficulty in the local approach arises from the well known fact that the field of a point charge diverges when the evaluation point for the field coincides with the field's source. In fact, this happens already for a static charge in empty (and flat) spacetime. Let the charge e be located on the $\hat{\mathbf{z}}$ axis in spherical coordinates. Decomposing the scalar potential Φ into Legendre polynomials, one finds that

$$\Phi(r,\vartheta)=e\sum_{\ell=0}^{\infty}\frac{r_<^{\ell}}{r_>^{\ell+1}}P_\ell(\cos\vartheta),$$

where $r_<$ ($r_>$) is the smaller (greater) of the r values of the source's location r_0 and the evaluation point. The (self) force acting on the charge is then given by the (average of the two one-sided) gradients of the potential. Specifically,

$$\mathbf{f}=-\frac{1}{2}e(\boldsymbol{\nabla}\Phi^+ +\boldsymbol{\nabla}\Phi^-)|_{\mathbf{r}=r_0\hat{\mathbf{z}}},$$

or

$$f_r=\sum_{\ell=0}^{\infty}\frac{e^2}{2r_0^2},$$

which clearly diverges. (The derivation of the last equation is given below.) In this illustration of the problem, of course, it is clear that the regularized, physical self-force vanishes: The force obviously cannot depend on where we choose to put the origin of our coordinate system. (Also, we have ample observational evidence that static isolated charges remain static.)

There is a long history of works on the self-force. (For reviews see, e.g., [2].) Recently, the analogous problem of calculation of self-forces in curved spacetimes (also for the gravitational case where the self-interaction pushes a body with finite mass off a geodesic) has gained much interest [3,4]. In this paper we shall make use of some of the techniques, which have been developed for self-interaction in curved spacetime, for the problem of interest. (Interestingly, there is a close link between electromagnetism in static gravitational fields and electromagnetism in matter. As is well known [5], Maxwell's equations in vacuum in static curved spacetime can be written as Maxwell's equations in flat spacetime with an effective nonuniform dielectric.) Specifically, we shall make use of the Quinn-Wald comparison axiom [6] and mode-sum regularization [4] in order to extract the physical, finite piece of the self-force.

The organization of this paper is as follows. In Sec. II we solve for the scalar potential, and obtain the modes of the bare force. This is, in fact, a standard exercise in electromagnetism [7]. Then, in Sec. III we regularize the self-force using two different approaches, and in Sec. IV we discuss the properties of our result.

II. DERIVATION OF THE BARE FORCE

Consider a static electric charge e in vacuum at radius r_0, outside an insulated sphere of radius R of uniform dielectric constant $\epsilon=1+\epsilon_0$, where $\epsilon_0>0$. Notice, that ϵ_0 is not the permittivity of free space, but rather $\epsilon_0=4\pi\chi_e$, where χ_e is the electric susceptibility. We place the charge e on the $\hat{\mathbf{z}}$ axis without loss of generality. This configuration is plotted in Fig. 1. Maxwell's equation in matter is

$$\nabla\cdot\mathbf{D}=4\pi\rho \tag{1}$$

where $\mathbf{D}=\epsilon\mathbf{E}$ is the displacement field, $\mathbf{E}$ is the electric field, and ρ is the density of free charges.

We assume that the distribution of the dielectric is spherically symmetric (although nonuniform). Specifically, we take $\epsilon=\epsilon(r)$. (Despite the uniformity of the sphere, the dielectric constant throughout space depends on r: it suffers a step-function discontinuity at the surface of the sphere.) In the usual spherical coordinates Eq. (1) becomes

$$\partial_r^2\Phi+\left(\frac{2}{r}+\frac{\partial_r\epsilon}{\epsilon}\right)\partial_r\Phi+\frac{1}{r^2\sin\vartheta}\partial_\vartheta(\sin\vartheta\,\partial_\vartheta\Phi)=-\frac{4\pi}{\epsilon(r)}\rho \tag{2}$$

where Φ is the scalar potential. We next decompose Eq. (2) into Legendre polynomials. That is, $\Phi(r,\vartheta)=\Sigma_\ell\phi^\ell(r)P_\ell(\cos\vartheta)$ and $\rho(r,\vartheta)=(e/4\pi)[\delta(r-r_0)/r_0^2]\Sigma_\ell(2\ell+1)P_\ell(\cos\vartheta)$. The radial equation then becomes

$$\partial_r^2\phi^\ell+\left(\frac{2}{r}+\frac{\partial_r\epsilon}{\epsilon}\right)\partial_r\phi^\ell-\frac{\ell(\ell+1)}{r^2}\phi^\ell=-(2\ell+1)e\frac{\delta(r-r_0)}{\epsilon(r_0)r_0^2}. \tag{3}$$

The boundary conditions for this equation are that ϕ^ℓ is continuous everywhere (which includes regularity at the origin and at infinity), but $\partial_r\phi^\ell$ is discontinuous at $r=R$ and at $r=r_0$. Specifically, these latter two conditions are that

$$\lim_{\sigma\to0^+}\epsilon(R-\sigma)\partial_r\phi^\ell(R-\sigma)=\lim_{\sigma\to0^+}\epsilon(R+\sigma)\partial_r\phi^\ell(R+\sigma)$$

(which comes from the continuity of the normal component of the displacement field at the surface of discontinuity), and

$$\lim_{\sigma\to0^+}[\partial_r\phi^\ell(r_0+\sigma)-\partial_r\phi^\ell(r_0-\sigma)]=-(2\ell+1)\frac{e}{r_0^2}$$

[which comes from integration of Eq. (3) across $r=r_0$, and using the continuity of ϕ^ℓ and ϵ there (the only discontinuity of ϵ is at $r=R$)].

The radial functions ϕ^ℓ then satisfy

$$\phi^\ell(r)=\begin{cases}A_\ell r^\ell & r<R \quad \text{(region I)}\\ B_\ell r^\ell+C_\ell r^{-\ell-1} & R<r<r_0 \quad \text{(region II)}\\ D_\ell r^{-\ell-1} & r>r_0 \quad \text{(region II)},\end{cases} \tag{4}$$

where the coefficients A_ℓ, B_ℓ, C_ℓ, and D_ℓ are found from the boundary conditions. We find that

$$A_\ell=\frac{2\ell+1}{2\ell+1+\ell\epsilon_0}\frac{e}{r_0^{\ell+1}}, \tag{5}$$

$$B_\ell=\frac{e}{r_0^{\ell+1}}, \tag{6}$$

$$C_\ell=-\frac{\ell}{2\ell+1+\ell\epsilon_0}\frac{R^{2\ell+1}}{r_0^{\ell+1}}e\epsilon_0, \tag{7}$$

$$D_\ell=er_0^\ell\left[1-\frac{\ell}{2\ell+1+\ell\epsilon_0}\left(\frac{R}{r_0}\right)^{2\ell+1}\epsilon_0\right], \tag{8}$$

such that the scalar potential Φ is given by

$$\Phi=\begin{cases} e\sum_{\ell=0}^{\infty}\frac{2\ell+1}{2\ell+1+\ell\epsilon_0}\frac{r^\ell}{r_0^{\ell+1}}P_\ell(\cos\vartheta) & r<R\\ \Phi_{\rm vac}-\sum_{\ell=0}^{\infty}\frac{\ell}{2\ell+1+\ell\epsilon_0}\frac{R^{2\ell+1}}{r_0^{\ell+1}}\frac{e\epsilon_0}{r^{\ell+1}}P_\ell(\cos\vartheta) & r>R.\end{cases} \tag{9}$$

Here,

$$\Phi_{\rm vac}\equiv\frac{e}{|\mathbf{r}-r_0\hat{\mathbf{z}}|}=e\sum_\ell\frac{r_<^\ell}{r_>^{\ell+1}}P_\ell(\cos\vartheta) \tag{10}$$

is the potential in the absence of a dielectric sphere.

The bare force $\mathbf{f}^{\rm bare}$ is found by $\mathbf{f}^{\rm bare}=-e\nabla\Phi$, evaluated at the location of the charge at $r=r_0$ and $\vartheta=0$. From symmetry, it is clear that any force is radial. We compute, then, the radial component of the force only. Differentiating Eq.

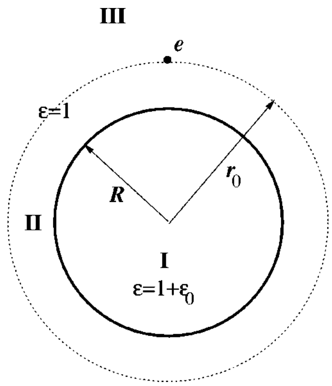

FIG. 1. An electric charge e at a distance r_0 from the center of a sphere of radius R. The sphere has a dielectric constant $\epsilon=1+\epsilon_0$, and outside the sphere the dielectric constant is unity. Region I is for $r<R$, region II for $R<r<r_0$, and region III for $r>r_0$.

(9) and using Eq. (10), we find that

$$f_r^{\rm bare}=\sum_{\ell=0}^{\infty} f_r^{\ell}$$

$$=\sum_{\ell=0}^{\infty}\left[\frac{e^2}{2r_0^2}-\frac{\ell(\ell+1)}{2\ell+1+\ell\epsilon_0}\left(\frac{R}{r_0}\right)^{2\ell+1}\frac{\epsilon_0 e^2}{r_0^2}\right], \quad (11)$$

where $f_r^{\ell}=-(e/2)\lim_{\sigma\to 0^+}[\partial_r\phi^{\ell}(r_0+\sigma)+\partial_r\phi^{\ell}(r_0-\sigma)]$. Clearly, Eq. (11) diverges. This comes as no surprise, as we have already mentioned that this divergence occurs already for a charge in empty space. In the following section we shall extract the physical, finite part of this infinite bare force.

III. REGULARIZATION OF THE BARE FORCE

In order to regularize the bare force (11), we make direct use of the Quinn-Wald comparison axiom, for which plausible arguments were given. The comparison axiom states the following (see [6] for more details): *Consider two points, P and $\tilde{P}$, each lying on timelike world lines in possibly different spacetimes that contain Maxwell fields $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ sourced by particles of charge e on the world lines. If the four-accelerations of the world lines at P and $\tilde{P}$ have the same magnitude, and if we identify the neighborhoods of P and $\tilde{P}$ via the exponential map such that the four-velocities and four-accelerations are identified via Riemann normal coordinates, then the difference between the electromagnetic forces f_μ and $\tilde{f}_\mu$ is given by the limit $x\to 0$ of the Lorentz force associated with the difference of the two fields averaged over a sphere at geodesic distance x from the world line at P, i.e.,*

$$f_\mu-\tilde{f}_\mu=\lim_{x\to 0} e\langle F_{\mu\nu}-\tilde{F}_{\mu\nu}\rangle_x u^\nu. \quad (12)$$

Here, we identify the "tilde" spacetime as that of a globally empty spacetime. Obviously, $\tilde{f}_\mu=0$. We emphasize that this axiom assumes a nearly trivial form for the case of interest: The local neighborhood of the particle in question and of a similar particle in a (globally)empty spacetime are identical. (It is only the far-away properties of spacetime—as represented by different dielectric constants—which are different for the two spacetimes.) Another remark is that we do not need to average here over directions, as the forces in our case are direction independent. Consider now Eq. (9) for the potential. Outside the dielectric sphere the potential Φ contains the vacuum potential $\Phi_{\rm vac}$ and a correction $\Delta\Phi$. We next use $\Phi_{\rm vac}$ to construct the fields $\tilde{F}_{\mu\nu}$. Applying the comparison axiom, we find that the self-force is given by

$$f_r=-\sum_{\ell=0}^{\infty}\frac{\ell(\ell+1)}{2\ell+1+\ell\epsilon_0}\left(\frac{R}{r_0}\right)^{2\ell+1}\frac{\epsilon_0 e^2}{r_0^2} \quad (13)$$

$$=-\frac{2}{3+\epsilon_0}\left(\frac{R}{r_0}\right)^3 {}_2F_1\left[3,\frac{3+\epsilon_0}{2+\epsilon_0};\frac{5+2\epsilon_0}{2+\epsilon_0};\left(\frac{R}{r_0}\right)^2\right]\frac{\epsilon_0 e^2}{r_0^2}, \quad (14)$$

${}_2F_1$ being the hypergeometric function. This result can be expressed in terms of the incomplete β function [8] as

$$f_r=-\frac{2}{2+\epsilon_0}\left(\frac{R}{r_0}\right)^{\epsilon_0/(2+\epsilon_0)} B_{R^2/r_0^2}\left(\frac{3+\epsilon_0}{2+\epsilon_0},-2\right)\frac{\epsilon_0 e^2}{r_0^2}. \quad (15)$$

Equation (14) [or Eq. (15)] is our main result. We were unable to find this result in the literature. (In view of the vastness of the literature on classical electromagnetism, our search in the literature is naturally incomplete.)

Before we analyze the properties of this result, let us derive it using a second method. Specifically, we use mode-sum regularization. (Note, that mode-sum regularization is based on the Quinn-Wald result for the self-force in curved spacetime, the latter being a consequence of the comparison axiom. In that sense, these two methods are not entirely independent. Here, however, we make direct use of the comparison axiom, which is necessary but not sufficient in order to derive the Quinn-Wald result.) Mode-sum regularization is described in Refs. [4]. In mode-sum regularization one finds two regularization functions, h_μ^{ℓ} and d_μ. The regularized self-force is given by

$$f_\mu=\sum_{\ell=0}^{\infty}(f_\mu^{\ell\ \rm bare}-h_\mu^{\ell})-d_\mu, \quad (16)$$

where d_μ is a finite valued function and h_μ^{ℓ} has the general form $h_r^{\ell}=a_r(\ell+\frac{1}{2})+b_r+c_r(\ell+\frac{1}{2})^{-1}$. One only needs the *local* properties of spacetime in order to determine these functions. As locally the charge is in empty space (it is removed from the dielectric sphere), it is clear that the regularization functions h_μ^{ℓ} and d_μ would be the same as in a globally empty spacetime. Indeed, it is easy to find the limit as $\ell\to\infty$ of the modes of the bare force. The modes of the radial component of the bare force (11) approach $e^2/(2r_0^2)$ as $\ell\to\infty$. As h_μ^{ℓ} must have the same asymptotic structure (as $\ell\to\infty$) as f_μ^{ℓ}, this implies that $h_r^{\ell}=e^2/(2r_0^2)$, identically the

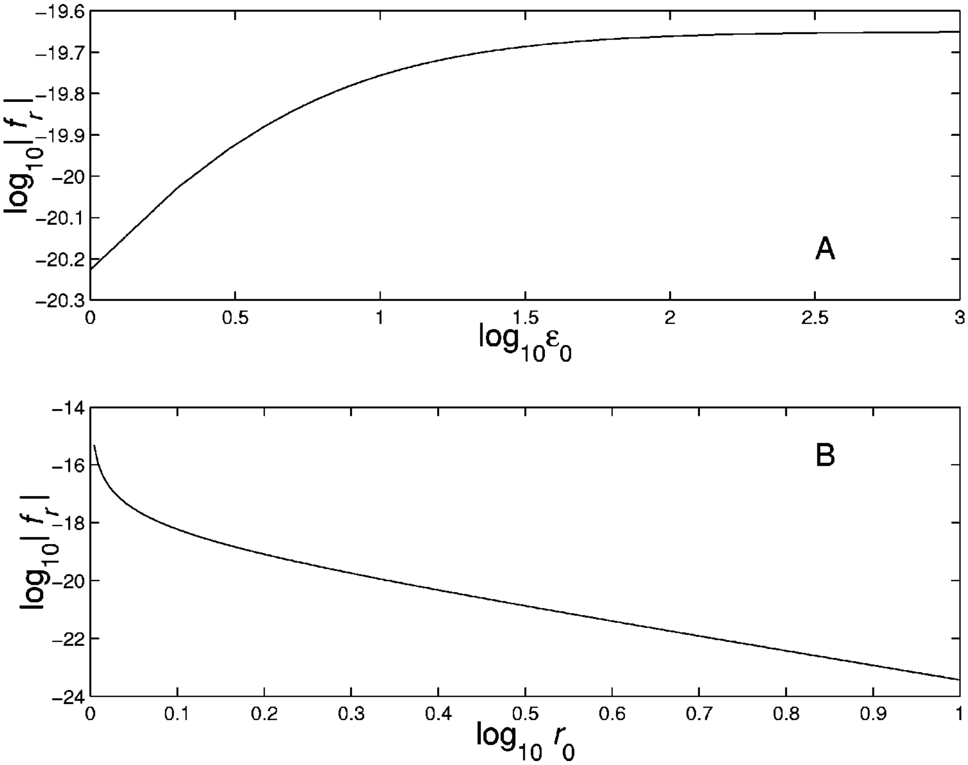

FIG. 2. The self-force on a free charge outside a dielectric sphere. The charge e is taken to be that of an electron, and the radius of the sphere is $R=1$ cm. Upper panel (A): The self-force as a function of ϵ_0, for $r_0=2$ cm. Lower panel (B): The self-force as a function of r_0 (in centimeters), for $\epsilon_0=10.7$.

same as in (globally)empty spacetime, in agreement with the previous reasoning. We similarly expect the function d_r to vanish, as it does in a globally empty flat spacetime. We justify this expectation *a posteriori* by demonstrating that this leads to the same expression as we received by using the comparison axiom. It then follows that the regularized self-force is given by $f_r=\Sigma_{\ell}[f_r^{\ell\ \mathrm{bare}}-e^2/(2r_0^2)]$, which agrees with Eq. (13).

IV. PROPERTIES OF THE RESULT

We found that the self-force on the charge e is given by Eq. (14). This is an attractive force, as indeed is expected. (The charge e polarizes the sphere such that there is an excess of oppositely charged induced charge on the sphere closer to the free charge. Hence the polarization charge acts to attract the free charge.) We can check our result in the limiting case of infinite dielectric, $\epsilon_0\to\infty$, which corresponds to the case of an uncharged, insulated, conducting sphere. In that limit our result becomes

$$f_r\to -2\left(\frac{R}{r_0}\right)^3 {}_2F_1\left[3,1;2;\left(\frac{R}{r_0}\right)^2\right]\frac{e^2}{r_0^2}=-\frac{2r_0^2-R^2}{(r_0^2-R^2)^2}\left(\frac{R}{r_0}\right)^3 e^2, \tag{17}$$

which is indeed the known result for an uncharged, insulated, conducting sphere [9]. The opposite extreme case is the limit as $\epsilon_0\to 0$. Linearizing our result in ϵ_0, we find that

$$f_r=-\frac{\sqrt{\pi}}{2}\left(\frac{R}{r_0}\right)^{3/2}\left[1-\left(\frac{R}{r_0}\right)^2\right]^{-3/2} \times P_{1/2}^{-3/2}\left(\frac{r_0^2+R^2}{r_0^2-R^2}\right)\frac{\epsilon_0 e^2}{r_0^2}+O(\epsilon_0^2), \tag{18}$$

which vanishes linearly with ϵ_0 as $\epsilon_0\to 0$. For any finite value of ϵ_0 the force is smaller in magnitude than in the case of a conducting sphere (17). This behavior is shown in Fig. 2(a), which plots the self-force as a function of ϵ_0 for fixed r_0. It can be seen that as $\epsilon_0\to\infty$, the full expression approaches the saturation value of the conducting sphere.

At very large distances ($r_0\gg R$), the self-force becomes

$$f_r=-\frac{2}{3+\epsilon_0}\left(\frac{R}{r_0}\right)^3\frac{\epsilon_0 e^2}{r_0^2}+O(r_0^{-7}), \tag{19}$$

which drops off as r_0^{-5}. This behavior can be seen from Fig. 2(b), which displays the self-force as a function of r_0 for fixed ϵ_0. We can check the validity of this limit by deriving Eq. (19) using the alternative picture. When $r_0\gg R$, we can treat the field of the charge e to the leading order in $1/r_0$ as a constant over the sphere. Let us take for simplicity the charge $e=\mathrm{sgn}(e)|e|$ to be on the positive $\hat{\mathbf{z}}$ axis. The electric field due to the free charge is $\mathbf{E}_0=-\mathrm{sgn}(e)|e|r_0^{-2}\hat{\mathbf{z}}$, and the polarization of the sphere then is just a constant inside the sphere, and is given by $\mathbf{P}=[3/(4\pi)][\epsilon_0/(3+\epsilon_0)]\mathbf{E}_0$. The dipole moment $\mathbf{p}$ can be obtained by a volume integral over $\mathbf{P}$. One finds then that $\mathbf{p}=[\epsilon_0/(3+\epsilon_0)]R^3\mathbf{E}_0$. The electric field $\mathbf{E}$ at $r_0\hat{\mathbf{z}}$ is found by $\mathbf{E}=[3\hat{\mathbf{z}}(\mathbf{p}\cdot\hat{\mathbf{z}})-\mathbf{p}]/r_0^3$, or $\mathbf{E}=-2pr_0^{-3}\mathrm{sgn}(e)\hat{\mathbf{z}}$, where $p\equiv|\mathbf{p}|=\epsilon_0 R^3 r_0^{-2}|e|/(3+\epsilon_0)$. The force on the charge e is simply $\mathbf{f}=e\mathbf{E}=-2pr_0^{-3}|e|\hat{\mathbf{z}}$, which is equal to the leading order term of Eq. (19).

When r_0 approaches R the self-force grows rapidly, and in the limit diverges. This is indeed expected: in this limit one has a point charge near a semiinfinite dielectric. The solution for the force is a classic image problem [9], which obviously diverges in the coincidence limit of the charge and its image. This divergence happens already in the case of the conducting sphere, as is evident from Eq. (17). In fact, we find that the self force diverges whenever the free charge is locally at a region with nonzero gradient of the dielectric constant.

We note that the magnitude of this self-force is not extremely small for realistic parameters. Take the charge e to be that of an electron of mass m_e, and the dielectric sphere to be made of Silicon, for which $\epsilon_0 = 10.7$ at room temperature and pressure, and take the sphere to be of radius 1 cm. In the gravitational field of the Earth, with gravitational acceleration of 980 cm/sec^2, the self-force equals the weight of the electron when $r_0 = 13.2$ cm.

ACKNOWLEDGMENTS

I thank Saul Teukolsky for pointing out to me that the main result of this paper can be expressed in terms of an incomplete β function, and Richard Price for discussions. This research was supported by the National Science Foundation through Grant No. PHY-9734871.

[1] L.M. Burko, Am. J. Phys. **68**, 456 (2000).

[2] T. Erber, Fortschr. Phys. **9**, 343 (1961); P. Pearle, in *Electromagnetism—Paths to Research*, edited by D. Teplitz (Plenum, New York, 1982); F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Reading, MA, 1965).

[3] L.M. Burko, Class. Quantum Grav. **17**, 227 (2000); Phys. Rev. Lett. **84**, 4529 (2000); C.O. Lousto, *ibid.* **84**, 5251 (2000); L. Barack and L.M. Burko, Phys. Rev. D **62**, 084040 (2000); L.M. Burko, Y.T. Liu, and Y. Soen, *ibid.* **63**, 024015 (2001); L.M. Burko and Y.T. Liu, *ibid.* **64**, 024006 (2001); S. Detweiler, Phys. Rev. Lett. **86**, 1931 (2001); M.J. Pfenning and E. Poisson, Phys. Rev. D **65**, 084001 (2002).

[4] L. Barack and A. Ori, Phys. Rev. D **61**, 061502 (2000); **64**, 124003 (2001); L. Barack, *ibid.* **64**, 084021 (2001).

[5] C. Møller, *The Theory of Relativity*, 2nd ed. (Oxford University Press, Oxford, 1972).

[6] T.C. Quinn and R.M. Wald, Phys. Rev. D **56**, 3381 (1997).

[7] See, e.g., W.K.H. Panofsky and M. Phillips, *Classical electricity and magnetism*, 2nd ed. (Addison-Wesley, Reading, MA, 1962).

[8] A. Erdélyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi, *Higher Transcendental Functions* (McGraw Hill, New York, 1953), Vol. I, P. 87.

[9] J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975).